Two Dimensional Random Variables We may often associate more than one numerical measure with the outcome of an experiment. For example, given a piece of steel we may be interested in its hardness, measured on some scale, or in its tensile strength (greatest longitudinal stress born before breaking apart). As another example, we may be interested to determine whether a relationship exists between educational level and yearly income for 40 year old women. The underlying idea here is that, given a sample space, we may define more than one random variable on this sample space and consider these random variables together.

Let S be a sample space. Also let X and Y be random variables defined on S. We call (X, Y) a 2 dimensional random vector, or a 2 dimensional random variable. Also, define an n dimensional random vector as $(X_1, X_2, \ldots, X_n)$.

Let (X, Y) be a 2 dimensional random vector. Then (X, Y) is said to be discrete if (X, Y) assumes a finite or a countably infinite number of values.

Let (X, Y) be a 2 dimensional random vector. Then with each possible outcome of (X, Y), say (x_i, y_j), associate a number, $f(x_i, y_j)$, which has the value $Prob(X = x_i \textit{ and } Y = y_j)$. Note that it is not necessarily true that $Prob(X = x_i \textit{ and } Y = y_j) = Prob(X = x_i)Prob(Y = y_j)$. This is a special case called independence which will be discussed below.

f is a probability function (probability mass function) for some random vector (X, Y) if

1. $f(x_i, y_j) \geq 0 \; \forall \; i, j$
2. $\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(x_i, y_j) = 1$

Example: Suppose you toss a fiar coin 3 times. Let the random variable X indicate how many HEADS you obtained on the first two tosses, and let Y indicate how many HEADS you obtained on the last two tosses. Calculate the joint probability mass function for X and for Y.

The following example is from Meyer's *Introductory Probability and Statistical Applications.* In what follows, let the random variable X represent the number of items produced by Line I and let the random variable Y represent the number of items produced by Line II.

	0	1	2	3	4	5	$\sum x_i$
0	0.00	0.01	0.03	0.05	0.07	0.09	
1	0.01	0.02	0.04	0.05	0.06	0.08	
2	0.01	0.03	0.05	0.05	0.05	0.06	
3	0.01	0.02	0.04	0.06	0.06	0.05	
$\sum y_i$							

1. Calculate the probability that Line II produces exactly 2 items.
2. Calculate the probability that Line I produces exactly 4 items.
3. Calculate the probability that Line I produces more than Line II.
4. Calculate the probability that total production exceeds 6.
5. Calculate the probability distribution of $Z = X + Y$.

Marginal Distributions

Let (X, Y) be a 2 dimensional random vector with a joint probability mass function $f(x_i, y_j)$. We may define *marginal distributions* for X and Y with probability mass functions

$$f_X(x_i) = \sum_j f(x_i, y_j)$$

and

$$f_Y(y_j) = \sum_i f(x_i, y_j)$$

What are the marginal densities from the previous example?

Conditional Distributions Recall that

$$Prob(A|B) = \frac{Prob(A \cap B)}{Prob(B)}$$

and calculate $Prob(X = x_i | Y = y_j)$.

Consider again the example from Meyer about the two production lines. Calculate

1. $Prob(Y = 0|X = 2)$
2. $Prob(Y = 1|X = 2)$
3. $Prob(Y = 2|X = 2)$
4. $Prob(Y = 3|X = 2)$

This idea allows us to define a new random variable: Given that Y has occured with outcome $Y = y_j$ define the random variable $X|Y = y_j$ with probability mass function

$$f_{X|Y=y_j}(x_i|y_j) = \frac{f_{X,Y}(x_i, y_j)}{f_{X,Y}(x_i, y_j)}$$

Note how our notation can provide a nice insight into the processes involved.

Introduction to Sums of Independent Random Variables Suppose that you and a friend are shooting foul shots. Suppose further that you have seen, from past observation, that you are successful on 60% of your shots, independently from one another. Your friend has done a similar analysis and is successful on 80%, again independently. You will take 2 shots, and your friend will take 3 shots. If your successes are independent of your friend's, calculate the probabilities that, when your successes are added, your total will be 0, 1, . . . 5.

Let's consider these ideas in a more formal way. Let X and Y be discrete random variables. The we say that X and Y are *independent* if

$$Prob(X = x_i, Y = y_i) \; = \; Prob(X = x_i) \cdot Prob(Y = y_i) \; \forall \; i, j$$

Clearly, the above example meets this condition. Notice how we may write the probability distribution for Z in a fairly simple way as follows.

There is a graphical way of picturing the above sum. Write down the probabilities associated with outcomes for X in a row. Below this, write down, in reverse order, the probabilities associated with the outcomes of Y in such a way that columns sum to the outcomes of Z. What we are constructing is call a convolution. We will have more to say about these later.