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Recall

- 2 Dimensional Random Variables (Random Vectors)
- Joint Distributions
- Marginal Distributions
- Conditional Distributions

We have the basic result
Discrete Random Variable

$$E[g(X,Y)] = \sum_i \sum_j g(x_i, y_j)p(x_i, y_j)$$

This allows us to define a measure of dependence between two random variables. Define the **covariance** of random variables X and Y as

$$cov[X,Y] \equiv E[(X - \mu_X)(Y - \mu_Y))]$$

Recall our previous example: toss a fair coin 3 times. Let the random variable X count the number of heads on the first 2 tosses and let the random variable Y count the number of heads on the last 2 tosses. Find the conditional distributions $Y|X = x$, and the covariance $cov[X,Y]$ of the two random variables.

Independent Random Variables Recall: events A and B are said to be independent if and only if

$$P[A \text{ and } B] = P[A]P[B].$$

Therefore, we will say that random variables X and Y are independent if and only if

$$P[X \leq a, Y \leq b] = P[X \leq a]P[Y \leq b].$$

It is easy to show that this implies that the joint distribution is the product of the marginal distributions. i.e.

$$f(x,y) = g(x)h(y), \quad f(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2), \quad etc.$$

for discrete random variables and for continuous random variables.

Suppose that X and Y are independent identically distributed (iid) discrete random variables with the following probability distribution.

x_i	0	1	2
$f(x_i)$	1/4	1/2	1/4

(i) Calculate $\mu = \mu_X = \mu_Y$.

(ii) Calculate the probability distribution of the random variable $Z = X + Y$.

(iii) Use your result from part (ii) to calculate $E(Z)$.

Marginal Distributions and the Joint Distribution Consider two scenarios. First, toss two fair coins. Let the random variable X be the number of heads appearing and let the random variable Y be the number of tails appearing. (Are X and Y probabilistically independent?)

1. Calculate the joint distribution of (X, Y).
2. Calculate the marginal distributions of X and Y.

Now let X be as above, but let Y also be the number of heads appearing.

1. Calculate the joint distribution of (X, Y).
2. Calculate the marginal distributions of X and Y.

Notice how the marginals are identical in each of these cases, but the joint distributions differ. From a knowledge of the joint distribution we may always calculate the marginals, but a knowledge of the marginals doesn't, in general, allow us to calculate the joint distribution. The obvious exception here is when X and Y are independent.

1. Prove that $E[X + Y] = E[X] + E[Y]$.

2. Prove that, if X and Y are independent, then $E[XY] = E[X]E[Y]$.

3. Prove that $cov(X, Y) = E[XY] - E[X]E[Y]$.