Moment Generating Functions: These functions will be very convenient tools and computational aides as we work with both continuous and discrete random variables. In particular, once we have computed the moment generating function for a random variable, the calculations of that random variable's mean and variance are greatly simplified. We will also make use of these functions to calculate the probability distributions of functions, especially sums, of random variables.

Denote by $M_X(t)$ the moment generating function of a random variable X and define this function as

$$M_X(t) \equiv E(e^{tX})$$

Recall that for discrete distributions with probability mass function $f(x_i)$

$$E(g(X)) = \sum_{x_i} g(x_i) f(x_i)$$

Therefore

$$E(e^{tX}) = \sum_{x_i} e^{tx_i} f(x_i)$$

Similarly, for continuous RV's

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Calculating Means: We can use the previously calculated moment generating functions to calculate the mean and variance (in fact, all the moments) of the corresponding random variables as follows. Take a first derivative of $M_X(t)$ with respect to t (assuming that we can bring the derivative into the summation)

$$\begin{aligned}\frac{d}{dt}M_X(t) &= \frac{d}{dt}\sum_{x_i} e^{tx_i}p(x_i) \\ &= \sum_{x_i}\frac{d}{dt}e^{tx_i}p(x_i) \\ &= \sum_{x_i} x_i e^{tx_i}p(x_i)\end{aligned}$$

Now evaluate the derivative at $t = 0$.

$$M'_X(0) = \sum_{x_i} x_i e^{0x_i}p(x_i) = \sum_{x_i} x_i p(x_i)$$

We see that the first derivative of the moment generating function evaluated at $t = 0$ is the expectation of the random variable! This result holds in the continuous case as well.

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Calculating Variances: Recall that we may compute the variance of a random variable as

$$V(X) = E(X^2) - E(X)^2$$

Take a second derivative of the moment generating function

$$\begin{aligned} \frac{d^2}{dt^2} M_X(t) &= \frac{d}{dt} \sum_{x_i} x_i e^{tx_i} p(x_i) \\ &= \sum_{x_i} x_i^2 e^{tx_i} p(x_i) \end{aligned}$$

and so

$$M_X''(0) = \sum_{x_i} x_i^2 e^{0x_i} p(x_i) = E(X^2)$$

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Reproductive Property of Gamma Random Variables

Using moment generating functions, show (for what conditions?) that the sum of two independent Gamma random variables is also a Gamma Random Variable.

(Reproductive Property of Normal Random Variables)
Using moment generating functions, show that the sum of two independent normal random variables is also a normal random variable.