We have now seen several examples of samples, populations, and their histograms. We know how to characterize these data sets with respect to central tendency (e.g. mean, median) and spread (e.g. standard deviation, inter-quartile range). We would now like to develop mathematical models which give rise to the sorts of distributions we actually see realized in data sets. To do this we will need some basics in set theory and probability. As we explore these ideas try to keep in mind the following situations for which we will develop models:

- A simple coin toss (also called a Bernoulli trial) with a possibly unfair or biased coin. We note Success (Heads) or Failure (Tails).
- A set of n simple coin tosses (also called a Binomial Random Variable) with a possibly unfair or biased coin. In this case we try to describe the probability of obtaining r successes on n independent trials.
- A waiting time scenario where we keep tossing a (possibly unfair or biased) coin until our first Heads/Success. (Alternately, we keep rolling a die until our first "Six"). This is modelled by a Geometric Random Variable.
- A waiting time scenario where we keep tossing a (possibly unfair or biased) coin until we have obtained r Heads/Successes. We try to describe the probability that this will take r trials, $r + 1$ trials, etc. This is modelled with a negative binomial distribution.

Simple Counting Arguments:

- Suppose you are putting together a Mr. Potato Head and you have in front of you 4 mouths, 3 noses, and 2 sets of eyes. How many different faces can you make?

Multistage Process: Let $= k_i$ be the number of ways to perform the i^{th} stage of an n stage process. Then the number of ways to perform the composite procedure is

$$N = \prod_{i=1}^{n} k_i$$

- Suppose you have 3 probability books: Feller, Olkin, and Billingsley. call them F, O, and B. In how many ways can you arrange them in order?

  Suppose now you add Kolmogorov (call it K). In how many ways can you arrange 2 of these 4 books? In how many ways can you arrange 3 of these 4 books?

  Permutations: The number of possible sequences (ordered arrangements) of n objects is $n!$. If the arrangement is to contain only k of these n objects, then the number of ways to form sequences is

$$_nP_k \equiv \frac{n!}{(n-k)!}$$

- Consider again the 4 books: F, O, B, and K. In how many ways may you select 2 of these books if the order of the selection is unimportant to you? That is, FK is the same grouping as KF.

  Combinations: The number of possible unordered groups which contain k out of n objects is simply the number of possible permutations divided by the number of ways to arrange k objects (eliminate the redundant groups).

$$_nC_k \equiv \binom{n}{k} \equiv \frac{n!}{(n-k)!k!}$$

An excellent reference for combinatorics is the text <u>*Applied Combinatorics*</u> *by Alan Tucker.*

Examples

1. How many (5 card) poker hands are possible?

2. How many of these hands are full houses?

3. You have 6 computers in front of you, call them A, B, C, D, E, and F. Suppose for a moment that 2 of these computers, say E and F, have defective monitors. You will turn on 3 of them on at random. In how many groups of three, selected as above, are there

   - no defective monitors?
   - one defective monitors?
   - two defective monitors?

4. There are 10 math majors, 15 engineering majors, and 5 students from other majors in a probability class. In how many ways can you select two students from this class in such a way that they are from different majors?

5. In how many distinct ways can you arrange the letters *b, o, o, k*? The letters *m, i, s, s, i, s, s, i, p, p, i*?

6. In how many ways can a set of n elements be arranged if there are n_1 objects of one type, n_2 objects of another type, and so on till n_k objects of the k^{th} type, where $\sum_{i=1}^{k} n_k = n$?

7. How many different binary numbers of length 7 are possible if 3 of the digits are *1* 's?

8. An urn holds 5 red balls and 3 green balls. How many different results are possible if you select 1 ball? Two balls? 3 balls?

9. An urn holds 15 red balls and 10 green balls. How many different results are possible if you select 5 balls?

10. In how many ways can you distribute a dozen eggs to 30 people?

11. In how many ways can a group of 8 people be divided into working groups of 2 people each?

12. In how many ways can a set of n elements be partitioned into k subsets with $n_1, n_2, \ldots, n_k$ elements in each subset?