*Pascal's Triangle, Combinations, and The Binomial Theorem*

*Mathematical Framework* We will need a few notions from set theory in order to make some progress.

**Definition 1** *A set is a collection of objects. These objects are called elements of the set. We typically will use an upper case letter, such as A, for a set and a lower case letter, such as a, for an element of a set. We write* $a \in A$.

**Definition 2** *A subset, say* $B$*, of a set* $A$*, is itself a set, every element of which also belongs to the set* $A$*. We write* $B \subseteq A$.

**Definition 3** *Two sets,* $A$ *and* $B$*, are said to be equal if each element of* $A$ *is also in* $B$*, and if each element of* $B$ *is also in* $A$*. We write* $B = A \quad \Leftrightarrow \quad B \subseteq A$ *and* $A \subseteq B$.

**Definition 4** *We will call the set* $C$ *the union of the sets* $A$ *and* $B$ *if every element of* $C$ *is also in* $A$ *or* $B$*. We write* $C = A \cup B$.

**Definition 5** *We will call the set* $C$ *the intersection of the sets* $A$ *and* $B$ *if every element of* $C$ *is in* $A$ *and is also in* $B$*. We write* $C = A \cap B$.

**Definition 6** *Suppose we have* $B \subset U$*. Then we define the complement of the set* $B$ *as the set of elements which are in* $B$ *but not in* $U$*. We write* $B'$ *or* $B^c$ *or* $\overline{B}$.

*Relationships*

- $A \cup A^c = S$.

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

*Venn Diagram Illustrations*

*Formal Proofs*

*Basic Notions of Probability:*

We can consider our course to develop methods of quantifying and characterizing uncertainty in various situations. For example, consider a coin toss with a fair coin. Toss the coin 20 times and, after each toss, calculate the *relative frequency* of heads. We will need a few terms.

- Experiment

- Outcome

- Sample Space

- Event

*Relative Frequency Notion of Probability:* Repeat an experiment many times. *(What does many mean?)* Let $E$ be an event associated with the experiment. Perhaps the most intuitive way to define the probability of an event $E$, which we will denote as $P(E)$, is as follows.

$$P(E) \equiv \lim_{n \to \infty} \frac{Number\ of\ times\ E\ occurs}{n}$$

There are two key ideas here: On the one hand, we must be able to reproduce the experiment as often as we like, under unchanging conditions. We are also assuming that the limit defined above exists. If these conditions are not met, then we must reconsider what we mean by the word *probability.*

*Classic Examples:*

- Tossing a fair coin and observing H or T.
- Rolling a fair die and observing 1, 2, 3, 4, 5, or 6.
- Observing the lifetime of a light bulb.
- Observing the lifetime of someone who is currently 40 years old and a non-smoker.
- Counting the number of a certain type of microorganism in a 10 `ml.` sample from a lake.

Notice that we may not directly attach a probability to the statement *We will have a colony on the moon in the year 2100* in a meaningful way with the above notion of probability.

*Axioms of Probability*

We wish to define the idea of the probability of an event in a formal way, consistent with the relative frequency notion of probability. That is, given a sample space $S$ we wish to assign numbers to subsets of $S$, which we will call events, in a consistent way. We may do this by considering what properties are necessary for consistency. Clearly we must have the following:

1. The number, or probability, associated with an event must be between zero and one. That is $0 \leq P(E) \leq 1$.
2. The number, or probability, associated with the sample space must be one. That is $P(S) = 1$.
3. If events $A$ and $B$ do not overlap, that is if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

We are led to the following.

**Definition 7** *Given a sample space $S$, we define a probability on $S$ as a function defined from subsets of $S$ into the real numbers such that, for any event in $S$, we have the following:*

1. $P(E) \geq 0$
2. $P(S) = 1$
3. $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

We have the following results:

- $A \subset B \Rightarrow P(A) \leq P(B)$.
- $P(A) \leq 1$.
- $P(\emptyset) = 0$.

Show that, in general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.